A Note on Variable Capital Utilization in Growth and Business Cycle Theory

Holger Strulik
Timo Trimborn

This Version: August 2013

Abstract. It was always considered to be a major achievement of modern business cycle economics that it was solidly grounded in neoclassical growth theory. Preserving this joint foundation, however, imposes discipline on the specification of models with variable capital utilization. In this note we show that conventional specifications of the depreciation cost of capital utilization and the labor supply elasticity, introduced in business cycle theory in order to generate a satisfactory amplification of shocks, entail counterfactual growth dynamics: the positive association between capital stock and GDP along the growth path turns negative. Across economies with access to the same technology, the economy with the lowest capital stock per capita is predicted to produce the highest output per capita. We compute for the involved elasticities lower and upper bounds between which these counterfactual dynamics are avoided.

Keywords: business cycles, economic growth, capital utilization, labor supply.

JEL: E20, E22, O10, O40.
Step 3: Restrict the model to be consistent with the growth facts.
(Prescott, 2006)


\[ I \]t is possible to unify business cycle and growth theory by insisting that business cycle models must be consistent with the empirical regularities of long-run growth.
(Rebelo, 2005)

1. Introduction

One of the main achievements of modern business cycle economics is the dissolution of the dichotomy between growth theory and business cycle theory that had characterized macroeconomic research before the 1980s. The two pillars of dynamic macro are now solidly based on the same foundation – the "neoclassical model". While this methodological point is explicitly made by the vanguard of business cycle theory (e.g. King and Rebelo, 1999; Rebelo, 2005; Prescott, 2006) it is probably also subscribed to by researchers working in the New-Keynesian-DSGE paradigm and studying extensions of the basic model with market frictions and behavioral aspects.

In this note we focus on one basic feature of the neoclassical growth model: the role of capital accumulation for growth. The neoclassical growth model explains why a growing economy expands with capital and output rising “in sync” and why, ceteris paribus, an economy less well endowed with capital is poorer. While these growth facts are well-known by every economist it is less well-known that the neoclassical growth model fails to predict adjustment dynamics of capital and income correctly once it is extended by variable capital utilization and endogenous labor supply, given that these additional mechanisms are modeled and calibrated as usually done in the business cycle literature.¹

In business cycle economics, on the other hand, it turned out to be a very successful idea to extend the basic neoclassical model with variable capital utilization. The extension, in conjunction with elastic labor supply, did not only add more realism to the model but it reduced also the dependance on implausible large shocks in order to generate realistic business cycle dynamics (see e.g. Greenwood et al., 1988; Burnside and Eichenbaum, 1996; King and Rebelo, 1999; Baxter and Farr, 2004).

This note is organized as follows. In the next section we set up the standard neoclassical business cycle model augmented by variable capital utilization. In Section 3 we discuss the basic

¹There exists little research in growth economics on the role of capital utilization and the available literature neglects variable labor supply such that the phenomenon addressed in this paper remained unnoticed (Dalgaard, 2003; Chatterjee, 2005; Boucekkine et al., 2009).
mechanisms driving the responses of macro-economic aggregates to a temporary shock (business cycle dynamics) and to a permanent shock (growth dynamics). In Section 4 we calibrate the model and show that the phenomenon of overreacting GDP is robustly observed for commonly used specification of the underlying elasticities in the business cycle literature. We compute lower and upper bounds between which the counterfactual dynamics are avoided.

2. The Model

We consider the standard neoclassical growth model with variable labor supply and variable capital utilization. For analytical convenience we use a setup in continuous time, as common in growth theory. Otherwise we closely follow the established business cycle literature such that the exposition can be brief (see e.g. King and Rebelo, 1999 for details).

The economy is populated by a continuum of households of measure one. Households derive utility from consumption $c$ and from leisure $1 - \ell$. Specifically, households maximize intertemporal utility $\int_0^\infty \left( \frac{c^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(1-\ell)^{1-\gamma} - 1}{1-\gamma} \right) e^{-\rho t} dt$, in which $\rho$ denotes the time preference rate and $1/\sigma$ and $1/\gamma$ are the intertemporal elasticity of substitution for consumption and leisure, respectively. Households face the budget constraint $\dot{a} = w\ell + ra - c$, in which $a$ denotes assets, $w$ the wage rate, and $r$ the interest rate. The first order conditions provide the Ramsey rule (1) and the optimal temporal allocation of consumption and leisure (2).

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(r - \rho) \quad (1)$$

$$w = \frac{\beta e^\sigma}{(1 - \ell)^\gamma}. \quad (2)$$

Firms employ labor at wage $w$ and rent capital $k$ at rate $r$. Additionally, firms can choose the intensity at which the stock of capital is used in production. Capital services $s$ are a product of capital stock and capital utilization $z$, i.e. $s = zk$. Firms produce output $y$ with services from physical capital $s$ and labor $\ell$ according to a Cobb-Douglas production technology $y = As^\alpha \ell^{1-\alpha}$ in which $A$ denotes total factor productivity. Capital depreciation $\delta$ depends on capital utilization $z$ according to a convex function. We follow the literature (e.g. Burnside and Eichenbaum, 1996, Baxter and Farr, 2005) and assume a constant elasticity of marginal depreciation costs. Specifically we set $\delta(z) = \delta_0 + \delta_1 z^\xi$ with $\xi > 1$, $\delta_0$, $\delta_1 \geq 0$.²

²There exists also an alternative approach that considers capital utilization in conjunction with endogenous capital maintenance (Licandro and Puch, 2000; Boucekkine and Ruiz-Tamarit, 2003). Here we focus on the more popular treatment of capital utilization in the business cycle literature.
Firms maximize profits, i.e. sales minus factor costs, \( A(zk)^{\alpha\ell} - (r + \delta(z))k - w\ell \). The first order conditions require that capital and labor are paid according to their marginal product, \( r = \alpha y/k - \delta(z) \) and \( w = (1 - \alpha)y/\ell \). In addition a condition for the optimal rate of capital utilization requires that the marginal product of capital utilization and the marginal cost of depreciation are equalized:

\[
\alpha A(zk)^{\alpha - 1}k^\ell - \alpha = \delta \xi z^\ell - 1.
\] (3)

In general equilibrium markets clear, output is used up for consumption, investment and depreciation, and \( a = k \). Inserting wages and interest rates into the household budget constraint we get the equation of motion for the aggregate capital stock.

\[
\dot{k} = A(zk)^{\alpha\ell} - c - \delta(z)k.
\] (4)

The equilibrium path of the economy is defined by equations (1) – (4).

3. Analysis

We focus the analysis on the role of variable capital utilization for the response of the economy to a temporary and a permanent shock of aggregate productivity. In line with the macro literature we interpret a temporary shock as an impulse to business cycle dynamics and a permanent shock as an impulse to growth dynamics.

3.1. Business Cycle Dynamics. The optimal rate of capital utilization is chosen in response to contemporaneous economic aggregates and involves no intertemporal considerations. In order to understand the adjustment mechanism it thus suffices to inspect condition (3). Higher productivity increases output and raises the left hand side of equation (3). Capital cannot change on impact and thus capital utilization increases to balance the equation.

The labor-leisure choice of households is driven by a wealth effect and a substitution effect. Higher temporary productivity raises the net present value of income. According to the wealth effect, a higher permanent income induces households to demand more consumption and leisure such that labor supply declines. The substitution effect occurs because higher productivity increases interest rates on impact and induces households to save more. In order to mitigate the drop in consumption households supply more labor during the first quarters after the shock. For ordinary business cycle shocks the substitution effect dominates and the model predicts that
output and hours worked are positively correlated during the business cycle.

The initially high labor supply induces a further increase of the capital utilization rate. There is an accelerator at work, amplifying the productivity shock. The amplification enables the model to produce the observed business cycles moments from reasonably sized productivity shocks (King and Rebelo, 1999, Chapter 6). The intuition for the amplifier can best be conveyed with reference to the equivalence result developed in Wen (1998). After substituting $z$ from (3) the production function can equivalently be written as $y = A_0 k^{\alpha} \ell^{1-\bar{\alpha}}$ with $A_0$ being a constant and $\bar{\alpha} \equiv \alpha(\xi - 1)/(\xi - \alpha) < \alpha$. Variable capital utilization works like an increase in the output elasticity of labor (i.e. the control variable that can jump on impact) and like a decrease in the output elasticity of capital (i.e. the predetermined state variable).

3.2. Growth Dynamics. Growth dynamics take place when current capital stock is below its long-run steady-state level. Such a situation may have been caused by a permanent productivity shock or, alternatively, by a natural disaster or war, which have pushed capital stock below steady-state level. From an analytical viewpoint these causes of below steady-state capital stock are observationally equivalent.

The wealth effect associated with a permanent increase of productivity induces higher demand for consumption and leisure and thus lower labor supply. The substitution effect resulting from the initially low capital stock and the associated high interest rates induces higher savings. Because households want to keep consumption smooth they raise their labor supply. Usually the substitution effect dominates and output and employment increase. But higher labor supply taken for itself is usually not strong enough to shift output in one move from the old steady state position above the new steady-state position. This means that the benchmark business cycle model is consistent with the growth facts and with the benchmark neoclassical growth model. It predicts that, ceteris paribus, an economy endowed with a small capital stock per capita produces less GDP per capita than a well-endowed one.

Variable capital utilization, however, has the power to destroy the mutual consistency of business cycle and growth dynamics. The reason is that the above steady-state level of the interest rate has an expansive effect on capital utilization. To see this rewrite condition (3) as

\[ \begin{align*}
  \text{The wealth effect dominates only if the intertemporal elasticity of substitution for consumption is implausible large (exceeding 1/\alpha).} \\
  \text{In principle, reverse adjustment dynamics are possible without variation in capital utilization. Yet this response requires an implausibly low capital share of income or an implausibly large labor supply elasticity (see Table 1 below). Another – equally implausible – case was reported in Campbell (1994, Table 5), namely when labor is non-separable in utility and the intertemporal elasticity of substitution is infinite.}
\end{align*} \]
\( \alpha y/k = \delta \xi z^\xi \). The left hand side is the marginal product of capital, which is decreasing in \( k \). Since capital stock after the productivity shock is below steady-state level, the marginal product of capital is relatively large and thus firms choose optimally a high rate of capital utilization to balance the equation. A high rate of capital utilization, in turn, increases the marginal product of labor and leads to a second round effect by further rising labor demand and output. It may then happen that output overreacts: an exogenous loss of capital induces a spontaneous boost of output above steady-state state level and subsequent adjustment from above, i.e. with negative growth rates, towards the steady state.

Empirical evidence on transitional growth dynamics can be obtained from countries experiencing an “exogenous” loss of capital stock caused, for example, by natural disaster or war. Cavallo and Noy (2011) survey the literature on disasters and GDP. Most of the more recent studies surveyed document a negative effect on GDP (see e.g. Raddatz, 2007, Strobl, 2009, Noy, 2009, and Hochrainer, 2009). Some studies distinguish between the immediate impact on GDP and the impact during the adjustment process and find GDP growth rates to be above trend in the years after the disaster (e.g. Strobl, 2009). Others find a prolonged period during which growth is below its long-run rate (e.g. Hochrainer, 2009). Never, however, is it found that GDP jumps above its long-run trend-level right after the disaster and then adjusts from above, that is with negative growth rates, towards the long-run trend.

There exist also an extensive literature on the economic effects of World War II. Alvarez-Cuadrado (2008) document that the European core countries lost between 60% and 80% of their capital stock between 1939 and 1945. Loss in human lives was much less (between 0.7% and 7.1% of population) such that capital stock per capita declined considerably. The resulting drop in GDP in these countries was between 41% and 54%\(^5\). Another strand of the literature investigates the reconstruction and catch-up phase following World War II. Dumke (1990) and Smolny (2000) argue that the high post-war growth rate of western European countries can be decomposed into technological catch-up and reconstruction (capital accumulation). Both authors assert that the latter explains a large part of GDP growth and emphasize the importance of neoclassical capital accumulation for higher growth during the catch-up phase. To summarize, the evidence shows that destruction of capital reduces GDP on impact and initiates a transition process during which GDP growth rates are higher compared to steady state. Nowhere is it documented that GDP jumps above its long-run trend-level right after a war and then adjusts

\(^5\)See also Harrison (1998), Barro (2006, table 1) and Barro and Ursua (2011, table C2).
from above with negative growth rates towards its steady state. Economies operating a lower capital stock, ceteris paribus, produce less GDP.

4. CALIBRATION AND RESULTS

4.1. Calibration. We follow closely King and Rebelo’s (1999) calibration of a “benchmark” business cycle model for the US economy. The only difference is that we normalize the rate of technological progress to zero. This is inconsequential for the results but convenient because it allows for a sensitivity analysis with respect to the elasticity of labor supply without the need to re-specify the utility function. We set the capital share \( \alpha \) to 1/3, normalize technology \( A \) to 1 and assume that the long-run interest rate is 6.5% annually, i.e. we set \( \rho = 0.065 \). We set the intertemporal elasticities for consumption and labor to 1 (i.e. assume log-log-utility) and adjust \( \beta \) such that individuals work 20 percent of their time (\( \ell^* = 0.2 \)). These values imply that the Frisch elasticity of labor supply, computed as \( \psi = (1 - \ell^*)/(\gamma \ell^*) \), is equal to 4.

The annual depreciation of capital at the steady-state is set to 10%. We normalize \( \delta_1 \) such that \( z^* = 1 \). Several related studies ignore the constant term \( \delta_0 \). For us, however, the introduction of \( \delta_0 \) in conjunction with the normalization of \( z^* \) is helpful in order to make results for different numerical specifications of the model comparable. To see this, recall that condition (3), for given \( \xi \), provides \( \delta_1 \) as a unique function of the capital output ratio. This means that \( \delta_1 \) is pinned down by the calibrated values of the depreciation rate and the real interest rate at the steady-state, \( \delta_1 = [r^* + \delta(z^*)]/\xi \). For the benchmark case with variable capital utilization we set \( \xi = 2 \), which is the point estimate of Basu and Kimball (1997) used by Baxter and Farr (2005). This implies \( \delta_1 = 0.0825 \) and \( \delta_0 = 0.0175 \). Later on, in the sensitivity analysis, we adjust \( \delta_0 \) in order to match the same capital output ratio and thus the same real interest rate as in the benchmark case. The case without variable capital utilization can be conceptualized as a case of infinitely high marginal adjustment costs, \( \xi = \infty \).

Results. We consider first the benchmark business cycle model with constant capital utilization (\( \xi = \infty \)). In order to study adjustment dynamics we inspect an economy resting at the steady-state and experiencing a ten percent loss of capital in period 0. We solve the non-linearized model numerically using the relaxation algorithm (Trimborn et al., 2008).

The left panel in Figure 1.A shows the phase diagram. The economy adjusts with rising capital stock and GDP: as households become wealthier and firms become better equipped with capital stock, output and income per capita increase. The other panels show the associated
Adjustment Dynamics after a 10% loss of capital. Left panel: phase diagram. Other panels: impulse responses.

We next investigate the benchmark model with variable capital utilization. The saddle-path of the phase diagram, shown in the left panel in Figure 1.B, is now downward sloping. The mutually amplifying forces of increasing capital utilization and increasing employment cause an adjustment of GDP from above. In other words, a disaster destroying a part of the economy’s capital stock raises output above steady-state level, after which GDP declines as the capital stock recovers. The impulse-responses of capital utilization and labor input associated with the “wrong” adjustment dynamics are not outrageously large and vary certainly within the range observed during ordinary business cycles (see e.g. Burnside and Eichenbaum, 1996; King and Rebelo, 1999). Moreover, the problem would not disappear by focussing on smaller shocks and smaller deviation of labor input and the utilization rate from steady-state. The saddle path is monotonously downward sloping.

6Considering very large shocks it is conceivable that labor supply and capital utilization are constrained by technically or institutionally determined upper limits. But this would just mean that adjustment dynamics from...
We next demonstrate robustness of the result. In order to do this in a condensed way we report for alternative $\sigma$ and alternative $\xi$ the threshold value of the Frisch elasticity of labor supply above which GDP adjusts from the “wrong” side after a loss of capital. Results are shown in Table 1. For example, the first entry in the table says that if $\sigma = 0.5$ and $\xi = 1.6$, then GDP adjust from above for values of the Frisch elasticity exceeding 2.2. There exists a lower bound of $\xi$ at 1.6, assumed when $\delta_0 = 0$, because we require that all calibrations match the same steady-state interest rate of 6.5%.

The general message of Table 1 is the following. When $\xi$ is small (meaning capital utilization is highly elastic) and $\sigma$ is large (consumption is inelastic, implying ceteris paribus a large response of labor supply), then a low Frisch elasticity is needed to prevent the “wrong” adjustment dynamics. This finding is intuitive: labor supply must be relatively inelastic in order to prevent an over-reaction of GDP if there is already a lot of elasticity in the economy because of $\sigma$ and $\xi$. The power of $\xi$ can also be assessed by inspecting the implied output elasticity of capital $\bar{\alpha}$. A value of $\xi = 2$ implies, for example, an $\bar{\alpha}$ of 0.2. In principle we could thus produce adjustment dynamics from the wrong side even for fixed capital utility utilization by assuming a counterfactually low income share of capital below 20 percent.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\xi = 1.6$</th>
<th>$\xi = 2$</th>
<th>$\xi = 3$</th>
<th>$\xi = 4$</th>
<th>$\xi = 10$</th>
<th>$\xi = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.5$</td>
<td>(2.2)</td>
<td>(4.0)</td>
<td>(26)</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$\sigma = 1$</td>
<td>0.9</td>
<td>1.3</td>
<td>2.4</td>
<td>3.3</td>
<td>6.8</td>
<td>16</td>
</tr>
<tr>
<td>$\sigma = 2$</td>
<td>0.5</td>
<td>0.7</td>
<td>1.1</td>
<td>1.3</td>
<td>1.9</td>
<td>2.6</td>
</tr>
<tr>
<td>$\sigma = 4$</td>
<td>0.3</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>1.1</td>
<td>1.3</td>
</tr>
</tbody>
</table>

The table shows the critical value of the Frisch elasticity of labor supply ($\phi$): For higher values of $\phi$ GDP is decreasing during transition, for lower values of $\phi$ GDP is increasing during transition. In parenthesis we report the implied output elasticity of capital $\bar{\alpha} \equiv \alpha / (\xi - \alpha)$.

For the case when $\sigma = 0.5$ results look not too bad. If $\xi = 2$, a Frisch elasticity below 4 would be sufficient in order to prevent “wrong” adjustment dynamics. Empirical evidence, however, suggests that $\sigma$ is likely larger than unity (see e.g. Chetty, 2006). Values of $\sigma$ between 2 and 4 are common in calibrations of growth and business cycle models (e.g. Barro and Sala-i-Martin, 2004; King and Rebelo, 1999). The benchmark business cycle model assumed $\sigma = 1$ and $\phi = 4$. In this case we would need a value of $\xi$ above 4, which would reduce the variability of capital utilization and the power of shock amplification significantly. The popular case of the “wrong direction” set in at the moment when the feasibility constraints are no longer binding.
indivisible labor (Rogerson, 1988) implies an infinite labor supply elasticity and thus adjustment
dynamics are always from the “wrong side” irrespective of $\xi$. Business cycle studies using capital
utilization in order to amplify shocks usually calibrate a high elasticity (a low $\xi$) to strengthen
the mechanism. Common are values between 1.1 and 2. Additionally $\sigma$ is usually set to a value
of unity or even larger. In these cases adjustment dynamics are from the “wrong” side for any
value of the Frisch elasticity calibrated in conventional business cycle economics.\(^7\)

5. Discussion

In this note we have shown that requiring consistency with neoclassical growth dynamics puts
severe constraints on the calibration in business cycle models with variable capital utilization.
This result is particularly important since there is (yet) little guidance on the specification of
capital utilization from the empirical literature. Here we have focussed on the perhaps most
popular treatment of capital utilization in business cycle economics. But the result is of course
of a general nature, disciplining the (future) modeling of depreciation-in-use models, including
various types of adjustment costs, maintenance, and other extensions. In order to predict that
economies with an inferior endowment of physical capital per capita produce less GDP per capita
than a superiorly endowed economy, capital utilization must not be “too elastic”. This in turn
limits the power of capital utilization and labor supply in the amplification of shocks, a feature
which is much needed in conventional business cycle economics (King and Rebelo, 1999).

Many studies employing variable capital utilization rely the numerical specification on Basu
and Kimball (1997) who obtained point estimates of about unity for the elasticity of marginal
depreciation with respect to utilization ($\xi − 1$ in our notation), and a 95% confidence interval
of about $[−0.2, 2.3]$, depending on specification.\(^8\) If we ignore the huge uncertainty involved and
take the point estimate for face value, then a $\xi$ of 2 and a $\sigma ≥ 1$ severely constraints the choice
of the Frisch elasticity on values below what is usually used in business cycle economics.

One potential way out of the dilemma is to move on to a broader conceptualization of capital.
It can be shown that counterfactual transitional dynamics by and large disappear if $\alpha$ equals
$2/3$. Employing Wen’s (1998) equivalence result this is hardly surprising. For example, for $\xi = 2$
the implied output elasticity of capital $\bar{\alpha}$ increases from mere 0.2 when $\alpha = 1/3$ to 0.5 when

\(^7\)For example, in the notation of the present paper, Burnside and Eichenbaum (1996) set $\sigma = 1$ and $\xi = 1.56$,
King and Rebelo (1999, Chapter 8) set $\sigma = 3$ and $\xi = 1.1$, and Baxter and Farr (2005) set $\sigma = 1$ and $\xi = 2$ in
their benchmark case.

\(^8\)Johnson finds a lower point estimate of about 0.4 and a much larger 95% confidence interval.
\( \alpha = 2/3 \). An output elasticity of 0.5 is large enough to ensure adjustment dynamics from the “right side” for reasonable values of the labor supply elasticity.

Assuming a high value of \( \alpha \) has further beneficial side effects. It increases the internal propagation of shocks because a larger share of production factors responds sluggishly. Moreover it is known, that it slows down the speed of convergence. But these are just preliminary thoughts on the direction of future research. An extensive analysis of the implied adjustment dynamics and impulse responses would require to disintegrate the labor supply decision, the process of human capital accumulation, and the intensity of factor use in production, an endeavor beyond the scope of this note.

**References**


Chatterjee, S., 2005, Capital utilization, economic growth and convergence, *Journal of Economic...*

Dalgaard, 2003; Idle capital and long-run productivity, Contributions to Macroeconomics 3, Article 6.


Rogerson, R., 1988, Indivisible labor, lotteries and equilibrium, Journal of Monetary Economics 21, 3-16.


